

Frobenius and Maschke type Theorems for Doi-Hopf modules and entwined modules revisited: a unified approach

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Abstract

We study when induction functors (and their adjoints) between categories of Doi-Hopf modules and, more generally, entwined modules are separable, resp. Frobenius. We present a unified approach, leading to new proofs of old results by the authors, as well as to some new ones. Also our methods provide a categorical explanation for the relationship between separability and Frobenius properties.

0 Introduction

Let H be a Hopf algebra, A an H -comodule algebra, and C an H -module coalgebra. Doi [17] and Koppinen [21] independently introduced unifying Hopf modules, nowadays usually called Doi-Koppinen-Hopf modules, or Doi-Hopf modules. These are at the same time A -modules, and C -comodules, with a certain compatibility relation. Modules, comodules, graded modules, relative Hopf modules, dimodules and Yetter-Drinfel'd modules are all special cases of Doi-Hopf modules. Properties of Doi-Hopf modules (with applications in all the above special cases) have been studied extensively in the literature. In [12], a Maschke type theorem is given, telling when the functor F forgetting the C -coaction reflects the splitness of an exact sequence, while in [13], it is studied when this functor is a Frobenius functor, this means that its right adjoint $\bullet \otimes C$ is at the same time a left adjoint.

The two problems look very different at first sight, but the results obtained in [12] and [13] indicate a relationship between them. The main result of [12] tells us that we have a Maschke Theorem for the functor F if C is finitely generated projective and there exists an A -bimodule C -colinear map $A \otimes C \rightarrow C^* \otimes A$ satisfying a certain normalizing condition. In [13], we have seen that F is Frobenius if C is finitely generated and projective and $A \otimes C$ and $C^* \otimes A$ are isomorphic as A -bimodules and C -comodules. This isomorphism can be described using a so-called H -integral, this is an element in $A \otimes C$ satisfying a certain centralizing condition. The same H -integrals appear also when one studies Maschke Theorems for G , the right adjoint of F (see

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[10]). This connection was not well understood at the time when [12] and [13] were written. The aim of this paper is to give a satisfactory explanation; in fact we will present a unified approach to both problems, and solve them at the same time. We will then apply the same technique for proving new Frobenius type properties: we will study when the other forgetful functor forgetting the A -action is Frobenius, and when a smash product $A \#_R B$ is a Frobenius extension of A and B . Let us first give a brief overview of new results obtained after [12] and [13].

1) In [10] and [11] the notion of *separable functor* (see [23]) is used to reprove (and generalize) the Maschke Theorem of [12]. In fact separable functors are functors for which a “functorial” type of Maschke Theorem holds. A key result due to Rafael [25] and del Río [26] tells us when a functor having a left (resp. right) adjoint is separable: the unit (resp. the counit) of the adjunction needs a splitting (resp. a cosplitting).

2) *Entwined modules* introduced in [2] in the context of noncommutative geometry generalize Doi-Hopf modules. The most interesting examples of entwined modules turn out to be special cases of Doi-Hopf modules, but, on the other hand, the formalism for entwined modules is more transparent than the one for Doi-Hopf modules. Many results for Doi-Hopf modules can be generalized to entwined modules, see e.g. [3], where the results of [12] and [13] are generalized to the entwined case.

3) In [8], we look at separable and Frobenius algebras from the point of view of nonlinear equations; also here we have a connection between the two notions: both separable and Frobenius algebras can be described using normalized solutions of the so-called FS-equation. But the normalizing condition is different in the two cases.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a covariant functor having a right adjoint G . From Rafael’s Theorem, it follows that the separability of F and G is determined by the natural transformations in $V = \underline{\text{Nat}}(GF, 1_{\mathcal{C}})$ and $W = \underline{\text{Nat}}(1_{\mathcal{D}}, FG)$. In the case where F is the functor forgetting the coaction, V and W are computed in [10]. In fact V and W can also be used to decide when G is a left adjoint of F . This is what we will do in Section 3; we will find new characterizations for (F, G) to be a Frobenius pair, and we will recover the results in [13] and [3]. In Section 4, we will apply the same technique to decide when the other forgetful functor is Frobenius, and in Section 5, we will study when the smash product of two algebras A and B is a Frobenius extension of A and B . This results in necessary and sufficient conditions for the Drinfel’d double of a finite dimensional Hopf algebra H (which is a special case of the smash product (see [14]) to be a Frobenius or separable over H .

We begin with a short section about separable functors and Frobenius pair of functors. We will explain our approach in the most classical situation: we consider a ring extension $R \rightarrow S$, and consider the restriction of scalars functor. We derive the (classical) conditions for an extension to be separable (i.e. the restriction of scalars functor is separable), split (i.e. the induction functor is separable), and Frobenius (i.e. restriction of scalars and induction functors form a Frobenius pair). We present the results in such a way that they can be extended to more general situations in the subsequent Sections.

Let us remark at this point that the relationship between Frobenius extensions and separable extensions is an old problem in the literature. A classical result, due to Eilenberg and Nakayama, tells us that, over a field k , any separable algebra is Frobenius. Several generalizations of this property exist; conversely, one can give necessary and sufficient conditions for a Frobenius extension to be separable (see [19, Corollary 4.1]). For more results and a history of this problem, we refer to [1], [20] and [18].

We use the formalism of entwined modules, as this turns out to be more elegant and more general than that of the Doi-Hopf modules; several left-right conventions are possible and there exists a dictionary between them. In [12] and [13], we have worked with right-left Doi-Hopf modules; here we will work in the right-right case, mainly because the formulae then look more natural.

Throughout this paper, k is a commutative ring. We use the Sweedler-Heyneman notation for comultiplications and coactions. For the comultiplication Δ on a coalgebra C , we write

$$\Delta(c) = c_{(1)} \otimes c_{(2)}.$$

For a right C -coaction ρ^r and a left C -coaction ρ^l on a k -module N , we write

$$\rho^r(n) = n_{[0]} \otimes n_{[1]}, \quad \rho^l(n) = n_{[-1]} \otimes n_{[0]}.$$

We omit the summation symbol Σ .

1 Separable functors and Frobenius pairs of functors

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a covariant functor. Recall [23] that F is called a *separable functor* if the natural transformation

$$\mathcal{F} : \text{Hom}_{\mathcal{C}}(\bullet, \bullet) \rightarrow \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)),$$

induced by F splits. From [25] and [26], we recall the following characterisation in the case F has an adjoint.

Proposition 1.1 *Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a right adjoint of F . Let $\eta : 1_{\mathcal{C}} \rightarrow GF$ and $\varepsilon : FG \rightarrow 1_{\mathcal{D}}$ be the unit and counit of the adjunction. Then*

- 1) *F is separable if and only if there exists $v \in V = \underline{\text{Nat}}(GF, 1_{\mathcal{C}})$ such that $v \circ \eta = 1_{\mathcal{C}}$, the identity natural transformation on \mathcal{C} .*
- 2) *G is separable if and only if there exists $\zeta \in W = \underline{\text{Nat}}(1_{\mathcal{D}}, FG)$ such that $\varepsilon \circ \zeta = 1_{\mathcal{D}}$, the identity natural transformation on \mathcal{D} .*

The separability of F implies a Maschke type Theorem for F : if a morphism $f \in \mathcal{C}$ is such that $F(f)$ has a one-sided inverse in \mathcal{D} , then f has a one-sided inverse in \mathcal{C} .

A pair of adjoint functors (F, G) is called a *Frobenius pair* if G is not only a right adjoint, but also a left adjoint of F . The following result can be found in any book on category theory: G is a left adjoint of F if and only if there exist natural transformations $v \in V = \underline{\text{Nat}}(GF, 1_{\mathcal{C}})$ and $\zeta \in W = \underline{\text{Nat}}(1_{\mathcal{D}}, FG)$ such that

$$F(v_M) \circ \zeta_{F(M)} = I_{F(M)}, \quad (1)$$

$$v_{G(N)} \circ G(\zeta_N) = I_{G(N)}, \quad (2)$$

for all $M \in \mathcal{C}, N \in \mathcal{D}$. In order to decide whether F or G is separable, or whether (F, G) is a Frobenius pair, one has to investigate the natural transformations $V = \underline{\text{Nat}}(GF, 1_{\mathcal{C}})$ and $W = \underline{\text{Nat}}(1_{\mathcal{D}}, FG)$. It often happens that the natural transformations in V and W are determined by single maps. In this Section we illustrate this in a classical situation and recover well-known results. In the coming Sections more general situations are considered.

Let $i : R \rightarrow S$ be a ring homomorphism, and let $F = \bullet \otimes_R S : \mathcal{M}_R \rightarrow \mathcal{M}_S$ be the induction functor. The restriction of scalars functor $G : \mathcal{M}_S \rightarrow \mathcal{M}_R$ is a right adjoint of F . The unit and counit of the adjunction are

$$\forall M \in \mathcal{M}_R, \quad \eta_M : M \rightarrow M \otimes_R S, \quad \eta_M(m) = m \otimes 1,$$

$$\forall N \in \mathcal{M}_S, \quad \varepsilon_N : N \otimes_R S \rightarrow N, \quad \varepsilon_N(n \otimes s) = ns.$$

Let us describe V and W . Given $v : GF \rightarrow 1_{\mathcal{M}_R}$ in V , it is not hard to prove that $\bar{v} = v_R : S \rightarrow R$ is left and right R -linear. Conversely, given an R -bimodule map $\bar{v} : S \rightarrow R$, a natural transformation $v \in V$ can be constructed by

$$\forall M \in \mathcal{M}_R, \quad v_M(m \otimes s) = m\bar{v}(s).$$

Thus we have

$$V \cong V_1 = \text{Hom}_{R,R}(S, R). \quad (3)$$

Now let $\zeta: 1_{\mathcal{M}_S} \rightarrow FG$ be in W . Then $e = \sum e^1 \otimes e^2 = \zeta_S(1) \in S \otimes_R S$ satisfies

$$\sum se^1 \otimes e^2 = \sum e^1 \otimes e^2 s, \quad (4)$$

for all $s \in S$. Conversely if e satisfies (4), then we can recover ζ

$$\forall N \in \mathcal{M}_S, \quad \zeta_N: N \rightarrow N \otimes_R S, \quad \zeta_N(n) = ne^1 \otimes e^2.$$

In the sequel, we omit the summation symbol, and write $e = e^1 \otimes e^2$, where it is understood implicitly that we have a summation. So we have

$$W \cong W_1 = \{e = e^1 \otimes e^2 \in S \otimes_R S \mid se^1 \otimes e^2 = e^1 \otimes e^2 s, \text{ for all } s \in S\}.$$

Combining all these data, we obtain the following result (cf. [23] for 1) and 2) and [8] for 3))

Theorem 1.2 *Let $i: R \rightarrow S$ be a ringhomomorphism, F the induction functor, and G the restriction of scalars functor.*

- 1) *F is separable if and only if there exists a conditional expectation, that is $\bar{v} \in V_1$ such that $\bar{v}(1) = 1$, i.e. S/R is a split extension.*
- 2) *G is separable if and only if there exists a separability idempotent, that is $e \in W_1$ such that $e^1 e^2 = 1$, i.e. S/R is a separable extension.*
- 3) *(F, G) is a Frobenius pair if and only if there exist $\bar{v} \in V_1$ and $e \in W_1$ such that*

$$\bar{v}(e^1)e^2 = e^1\bar{v}(e^2) = 1. \quad (5)$$

Theorem 1.2 2) explains the terminology for separable functors. Theorem 1.2 3) implies the following

Corollary 1.3 *We use the same notation as in Theorem 1.2. If (F, G) is a Frobenius pair, then S is finitely generated and projective as a (right) R -module.*

Proof. For all $s \in S$, we have $s = se^1\bar{v}(e^2) = e^1\bar{v}(e^2s)$, hence $\{e^1, \bar{v}(e^2\bullet)\}$ is a dual basis for S as a right R -module. \square

We have a similar property if G is separable. For the proof we refer to [24].

Proposition 1.4 *With the same notation as in Theorem 1.2, if S is an algebra over a commutative ring R , S is projective as an R -module and G is separable, then S is finitely generated as an R -module.*

Using other descriptions of V and W , we find other criteria for F and G to be separable or for (F, G) to be a Frobenius pair. Let $\text{Hom}_R(S, R)$ be the set of right R -module homomorphisms from S to R . $\text{Hom}_R(S, R)$ is an (R, S) -bimodule:

$$(rfs)(t) = rf(ts), \quad (6)$$

for all $f \in \text{Hom}_R(S, R)$, $r \in R$ and $s, t \in S$.

Proposition 1.5 *Let $i: R \rightarrow S$ be a ringhomomorphism and use the notation introduced above. Then*

$$V = \underline{\text{Nat}}(GF, 1_c) \cong V_2 = \text{Hom}_{R, S}(S, \text{Hom}_R(S, R)).$$

Proof. Define $\alpha_1 : V_1 \rightarrow V_2$ as follows: for $\bar{v} \in V_1$, let $\alpha_1(\bar{v}) = \bar{\phi} : S \rightarrow \text{Hom}_R(S, R)$ be given by

$$\bar{\phi}(s)(t) = \bar{v}(st).$$

Given $\bar{\phi} \in V_2$, put

$$\alpha^{-1}(\bar{\phi}) = \bar{\phi}(1).$$

We invite the reader to verify that α_1 and α_1^{-1} are well-defined and that they are inverses of each other. \square

Proposition 1.6 *Let $i : R \rightarrow S$ be a ringhomomorphism and assume that S is finitely generated and projective as a right R -module. Then, with the notation introduced above,*

$$W = \underline{\text{Nat}}(1_{\mathcal{D}}, FG) \cong W_2 = \text{Hom}_{R,S}(\text{Hom}_R(S, R), S).$$

Proof. Let $\{s_i, \sigma_i \mid i = 1, \dots, m\}$ be a finite dual basis of S as a right R -module. Then for all $s \in S$ and $f \in \text{Hom}_R(S, R)$,

$$s = \sum_i s_i \sigma_i(s) \quad \text{and} \quad f = \sum_i f(s_i) \sigma_i.$$

Define $\beta_1 : W_1 \rightarrow W_2$ by $\beta_1(e) = \phi$, with

$$\phi(f) = f(e^1)e^2,$$

for all $f \in \text{Hom}_R(S, R)$. To show that ϕ is a left R -linear and right S -linear map, take any $r \in R$, $s \in S$ and compute

$$\begin{aligned} \phi(fs) &= f(se^1)e^2 = f(e^1)e^2s = \phi(f)s, \\ \phi(rf) &= \sum r f(e^1)e^2 = r\phi(f). \end{aligned}$$

Conversely, for $\phi \in W_2$ define

$$\beta_1^{-1}(\phi) = e = \sum_i s_i \otimes \phi(\sigma_i).$$

Then for all $s \in S$

$$\begin{aligned} \sum_i s_i \otimes \phi(\sigma_i)s &= \sum_i s_i \otimes \phi(\sigma_i s) \\ &= \sum_{i,j} \sum_i s_i \otimes \phi(\sigma_i(ss_j)\sigma_j) \\ &= \sum_{i,j} \sum_i s_i \sigma_i(ss_j) \otimes \phi(\sigma_j) \\ &= \sum_j ss_j \otimes \sigma_j, \end{aligned}$$

i.e., $e \in W_1$. Finally, β_1 and β_1^{-1} are inverses of each other since

$$\beta_1(\beta_1^{-1}(\phi))(f) = \beta_1(\sum_i s_i \otimes \phi(\sigma_i))(f) = \sum_i f(s_i)\phi(\sigma_i) = \sum_i \phi(f(s_i)\sigma_i) = \phi(f),$$

$$\beta_1^{-1}(\beta_1(e)) = \sum_i s_i \otimes \beta_1(e)(\sigma_i) = \sum_i s_i \otimes \sigma_i(e^1)e^2 = \sum_i s_i \sigma_i(e^1) \otimes e^2 = e.$$

\square

Theorem 1.7 Let $i : R \rightarrow S$ be a ringhomomorphism. We use the notation introduced above.

1) $F : \mathcal{M}_R \rightarrow \mathcal{M}_S$ is separable if and only if there exists $\bar{\phi} \in V_2$ such that $\bar{\phi}(1)(1) = 1$.

2) Assume that S is projective as a right R -module. Then G is separable if and only if S is finitely generated as a right R -module and there exists $\phi \in W_2$ such that

$$\sum_i s_i \phi(\sigma_i) = 1.$$

3) (F, G) is a Frobenius pair if and only if S is finitely generated and projective as a right R -module, and $\text{Hom}_R(S, R)$ and S are isomorphic as (R, S) -bimodules, i.e. S/R is Frobenius.

Proof. The result is a translation of Theorem 1.2 in terms of V_2 and W_2 , using Proposition 1.4 (for 2)) and Corollary 1.3 (for 3)). We prove one implication of 3). Assume that (F, G) is a Frobenius pair. From Corollary 1.3, we know that S is finitely generated and projective. Let $v \in V_1$ and $e \in W_1$ be as in part 3) of Theorem 1.2, and take $\bar{\phi} = \alpha_1(\bar{v}) \in V_2$, $\phi = \beta_1(e) \in W_2$. For all $f \in \text{Hom}_R(S, R)$ and $s \in S$, we have

$$(\bar{\phi} \circ \phi)(f)(s) = \bar{v}(\phi(f)s) = \bar{v}(f(e^1)e^2s) = f(e^1)\bar{v}(e^2s) = f(se^1)\bar{v}(e^2) = f(se^1\bar{v}(e^2)) = f(t)$$

and

$$(\phi \circ \bar{\phi})(s) = \bar{\phi}(s)(e_1)e_2 = \bar{v}(se^1)e^2 = \bar{v}(e^1)e^2s = s.$$

□

2 Entwined modules and Doi-Hopf modules

Let k be a commutative ring, A a k -algebra, C a (flat) k -coalgebra, and $\psi : C \otimes A \rightarrow A \otimes C$ a k -linear map. We use the following notation, inspired by the Sweedler- Heyneman notation:

$$\psi(c \otimes a) = a_\psi \otimes c^\psi.$$

If the map ψ occurs more than once in the same expression, we also use Ψ or Ψ' as summation indices, i.e.,

$$\psi(c \otimes a) = a_\Psi \otimes c^\Psi = a_{\Psi'} \otimes c^{\Psi'}.$$

(A, C, ψ) is called a (right-right) *entwining structure* if the following conditions are satisfied for all $a \in A$ and $c \in C$,

$$(ab)_\psi \otimes c^\psi = a_\psi b_\psi \otimes c^{\psi\psi}, \quad (7)$$

$$\varepsilon_C(c^\psi) a_\psi = \varepsilon_C(c) a, \quad (8)$$

$$a_\psi \otimes \Delta_C(c^\psi) = a_{\psi\psi} \otimes c_{(1)}^\psi \otimes c_{(2)}^\psi, \quad (9)$$

$$1_\psi \otimes c^\psi = 1 \otimes c. \quad (10)$$

A k -module M together with a right A -action and a right C -coaction satisfying the compatibility relation

$$\rho^r(ma) = m_{[0]} a_\psi \otimes m_{[1]}^\psi \quad (11)$$

is called an *entwined module*. The category of entwined modules and A -linear C -colinear maps is denoted by $\mathcal{C} = \mathcal{M}(\psi)_A^C$. An important class of examples comes from *Doi-Koppinen-Hopf structures*. A (right-right) Doi-Koppinen-Hopf structure consists of a triple (H, A, C) , where H is a k -bialgebra, A a right H -comodule algebra, and C a right H -module coalgebra. Consider the map $\psi : C \otimes A \rightarrow A \otimes C$ given by

$$\psi(c \otimes a) = a_{[0]} \otimes ca_{[1]}.$$

Then (A, C, ψ) is an entwining structure, and the compatibility relation (11) takes the form

$$\rho^r(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]}. \quad (12)$$

A k -module with an A -action and a C -coaction satisfying (12) is called a *Doi-Koppinen-Hopf module* or a *Doi-Hopf module*. Doi-Koppinen-Hopf modules were introduced independently by Doi in [17] and Koppinen in [21]. Properties of these modules were studied extensively during the last decade, see e.g. [10], [11], [12], [13], [14], [15]. Another class of entwining structures is related to coalgebra Galois extensions, see [6] for details. Entwining structures were introduced in [7]. Many properties of Doi-Hopf modules can be generalized to entwined modules (see e.g. [3], [4]). Although the most studied examples of entwined modules (graded modules, Yetter-Drinfel'd modules, dimodules, Hopf modules) are special cases of Doi-Hopf modules, their properties can be formulated more elegantly in the language of entwined modules. The functor $F : {}^C\mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}_A$ forgetting the C -coaction has a right adjoint $G = \bullet \otimes C$. The structure on $G(M) = M \otimes C$ is given by the formulae

$$\rho^r(m \otimes c) = m \otimes c_{(1)} \otimes c_{(2)}, \quad (13)$$

$$(m \otimes c)a = ma_\psi \otimes c^\psi. \quad (14)$$

For later use, we list the unit and counit natural transformations describing the adjunction,

$$\rho : 1_C \rightarrow GF \text{ and } \varepsilon : FG \rightarrow 1_{\mathcal{M}_A},$$

$$\rho_M : M \rightarrow M \otimes C, \quad \rho_M(m) = \sum m_{[0]} \otimes m_{[1]},$$

$$\varepsilon_N = I_N \otimes \varepsilon_C : N \otimes C \rightarrow N.$$

In particular, $A \otimes C \in {}^C\mathcal{M}(\psi)_A^C$. $A \otimes C$ is also a left A -module, the left A -action is given by $a(b \otimes c) = ab \otimes c$. This makes $A \otimes C$ into an object of ${}_A\mathcal{M}(\psi)_A^C$, the category of entwined modules with an additional left A -action that is right A -linear and right C -colinear.

The other forgetful functor $G' : {}^C\mathcal{M}(\psi)_A^C \rightarrow {}^C\mathcal{M}$ has a left adjoint $F' = \bullet \otimes A$. The structure on $F'(N) = N \otimes A$ is now given by

$$\rho^r(n \otimes a) = n_{[0]} \otimes a_\psi \otimes n_{[1]}^\psi, \quad (15)$$

$$(n \otimes a)b = n \otimes ab. \quad (16)$$

The unit and counit of the adjunction are

$$\mu : F'G' \rightarrow 1_C \text{ and } \eta : 1_{{}^C\mathcal{M}} \rightarrow G'F',$$

$$\mu_M : M \otimes A \rightarrow A, \quad \mu_M(m \otimes a) = ma,$$

$$\eta_N : N \rightarrow N \otimes A, \quad \eta_N(n) = n \otimes 1.$$

In particular $G'(C) = C \otimes A \in {}^C\mathcal{M}(\psi)_A^C$. The map $\psi : C \otimes A \rightarrow A \otimes C$ is a morphism in ${}^C\mathcal{M}(\psi)_A^C$. $C \otimes A$ is also a left C -comodule, the left C -coaction being induced by the comultiplication on C . This coaction is right A -linear and right C -colinear, and thus $C \otimes A$ is an object of ${}^C\mathcal{M}(\psi)_A^C$, the category of entwined modules together with a right A -linear right C -colinear left C -coaction.

3 The functor forgetting the coaction

Let (A, C, ψ) be a right-right entwining structure, $F : \mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}_A$ the functor forgetting the coaction, and $G = \bullet \otimes C$ its adjoint. In [13] necessary and sufficient conditions for (F, G) to be a Frobenius pair are given (in the Doi-Hopf case; the results were generalized to the entwining case in [3]), under the additional assumption that C is projective as a k -module. In this Section we give an alternative characterization that also holds if C is not necessarily projective, and we find a new proof of the results in [13] and [3]. The method of proof is the same as in Section 1, i.e., based on explicit descriptions of V and W . These descriptions can be found in [10], [11] and [4] in various degrees of generality. To keep this paper self-contained, we give a sketch of proof. We first investigate $V = \underline{\text{Nat}}(GF, 1_C)$. Let V_1 be the k -module consisting of all k -linear maps $\theta : C \otimes C \rightarrow A$ such that

$$\theta(c \otimes d)a = a_{\psi\psi}\theta(c^\Psi \otimes d^\Psi), \quad (17)$$

$$\theta(c \otimes d_{(1)}) \otimes d_{(2)} = \theta(c_{(2)} \otimes d)_{\psi} \otimes c_{(1)}^\Psi. \quad (18)$$

Proposition 3.1 *The map $\alpha : V \rightarrow V_1$ given by $\alpha(v) = \theta$, with*

$$\theta(c \otimes d) = (I_A \otimes \varepsilon_C)(v_{A \otimes C}(1_A \otimes c \otimes d)), \quad (19)$$

is an isomorphism of k -modules. The inverse $\alpha^{-1}(\theta) = v$ is defined as follows: $v_M : M \otimes C \rightarrow M$ is given by

$$v_M(m \otimes c) = m_{[0]}\theta(m_{[1]} \otimes c). \quad (20)$$

Proof. Consider $\underline{v} = v_{A \otimes C}$ and $\bar{v} = v_{C \otimes A}$. Due to the naturality of v and (7) there is a commutative diagram

$$\begin{array}{ccccc} C \otimes A \otimes C & \xrightarrow{\bar{v}} & C \otimes A & \xrightarrow{\varepsilon_C \otimes I_A} & A \\ \psi \otimes I_C \downarrow & & \psi \downarrow & & I_A \downarrow \\ A \otimes C \otimes C & \xrightarrow{\underline{v}} & A \otimes C & \xrightarrow{I_A \otimes \varepsilon_C} & A \end{array}$$

Write $\bar{\lambda} = (I_A \otimes \varepsilon_C) \circ \bar{v}$ and $\underline{\lambda} = (\varepsilon_C \otimes I_A) \circ \underline{v}$. Then it follows that

$$\theta(c \otimes d) = \bar{\lambda}(c \otimes 1 \otimes d) = \underline{\lambda}(1 \otimes c \otimes d).$$

We have seen before that $A \otimes C \in {}_A\mathcal{M}(\psi)_A^C$. It is easy to prove that $GF(A \otimes C) = A \otimes C \otimes C \in {}_A\mathcal{M}(\psi)_A^C$ - the left A -action is induced by the multiplication in A - and \underline{v} is a morphism in ${}_A\mathcal{M}(\psi)_A^C$. Thus \underline{v} and $\underline{\lambda}$ are left and right A -linear, and

$$\begin{aligned} \theta(c \otimes d)a &= \underline{\lambda}(1 \otimes c \otimes d)a = \underline{\lambda}(a_{\psi\psi} \otimes c^\Psi \otimes d^\Psi) \\ &= a_{\psi\psi}\underline{\lambda}(1 \otimes c^\Psi \otimes d^\Psi) = a_{\psi\psi}\theta(c^\Psi \otimes d^\Psi), \end{aligned}$$

proving (17). To prove (18), we first observe that $C \otimes A, GF(C \otimes A) = C \otimes C \otimes A \in {}^C\mathcal{M}(\psi)_A^C$, the left C -coaction is induced by comultiplication in C in the first factor. Also \bar{v} is a morphism in ${}^C\mathcal{M}(\psi)_A^C$, and we conclude that \bar{v} is left and right C -colinear. Take $c, d \in C$, and put

$$\bar{v}(c \otimes d \otimes 1) = \sum_i c_i \otimes a_i.$$

Writing down the condition that \bar{v} is left C -colinear, and then applying ε_C to the second factor, we find that

$$c_{(1)} \otimes \theta(c_{(2)} \otimes d) = \sum_i c_i \otimes a_i = \bar{v}(c \otimes d \otimes 1). \quad (21)$$

Since \bar{v} is also right C -colinear,

$$\bar{v}(c \otimes 1 \otimes d_{(1)}) \otimes d_{(2)} = \sum_i c_{i(1)} \otimes a_{i\psi} \otimes c_{i(2)}^\psi$$

and, applying ε_C to the second factor, we find

$$\theta(c \otimes d_{(1)}) \otimes d_{(2)} = \psi\left(\sum_i c_i \otimes a_i\right), \quad (22)$$

and (18) follows from (21) and (22). This proves that there is a well-defined map $\alpha : V \rightarrow V_1$.

To show that the map α^{-1} defined by (20) is well-defined, take $\theta \in V_1$, $M \in \mathcal{C}$, and let v_M be given by (20). It needs to be shown that $v_M \in \mathcal{C}$, i.e., v_M is right A -linear and right C -colinear, and that v is a natural transformation. The right A -linearity follows from (17), and the right C -colinearity from (18). Given any morphism $f : M \rightarrow N$ in \mathcal{C} , one easily checks that for all $m \in M$ and $c \in C$

$$v_N(f(m) \otimes c) = f(m_{[0]})\theta(m_{[1]} \otimes c) = f(m_{[0]}\theta(m_{[1]} \otimes c)) = f(v_M(m \otimes c)),$$

i.e., v is natural. The verification that α and α^{-1} are inverses of each other is left to the reader. \square

Now we give a description of $W = \underline{\text{Nat}}(1_{\mathcal{M}_A}, FG)$. Let

$$W_1 = \{z \in A \otimes C \mid az = za, \text{ for all } a \in A\},$$

i.e., $z = \sum_l a_l \otimes c_l \in W_1$ if and only if

$$\sum_l aa_l \otimes c_l = \sum_l a_l a_\psi \otimes c_l^\psi. \quad (23)$$

Proposition 3.2 *Let (A, C, ψ) be a right-right entwining structure. Then there is an isomorphism of k -modules $\beta : W \rightarrow W_1$ given by*

$$\beta(\zeta) = \zeta_A(1). \quad (24)$$

The inverse of β is $\beta^{-1}(\sum_l a_l \otimes c_l) = \zeta$, with $\zeta_N : N \rightarrow N \otimes C$ given by

$$\zeta_N(n) = \sum_l na_l \otimes c_l. \quad (25)$$

Proof. We leave the details to the reader; the proof relies on the fact that ζ_A is left and right A -linear. \square

In [10], Propositions 3.1 and 3.2 are used to determine when the functor F and its adjoint G are separable.

Theorem 3.3 *Let $F : \mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}_A$ be the forgetful functor, and $G = \bullet \otimes C$ its adjoint. F is separable if and only if there exists $\theta \in V_1$ such that*

$$\theta \circ \Delta_C = \varepsilon_C.$$

G is separable if and only if there exists $z = \sum_l a_l \otimes c_l \in W_1$ such that

$$\sum_l \varepsilon_C(c_l) a_l = 1.$$

Proof. This follows immediately from Propositions 1.1, 3.1 and 3.2 \square

Next we show that the fact that (F, G) is a Frobenius pair is also equivalent to the existence of $\theta \in V_1$ and $z \in W_1$, but now satisfying different normalizing conditions.

Theorem 3.4 *Let $F : \mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}_A$ be the forgetful functor, and $G = \bullet \otimes C$ its adjoint. Then (F, G) is a Frobenius pair if and only if there exist $\theta \in V_1$ and $z = \sum_l a_l \otimes c_l \in W_1$ such that the following normalizing condition holds, for all $d \in C$:*

$$\epsilon_C(d)1 = \sum_l a_l \theta(c_l \otimes d) \quad (26)$$

$$= \sum_l a_l \psi \theta(d^\psi \otimes c_l). \quad (27)$$

Proof. Suppose that (F, G) is a Frobenius pair. Then there exist $v \in V$ and $\zeta \in W$ such that (1-2) hold. Let $\theta = \alpha(v) \in V_1$, and $z = \sum_l a_l \otimes c_l = \beta(\zeta) \in W_1$. Then (1) can be rewritten as

$$v_M(\sum_l m a_l \otimes c_l) = \sum m_{[0]} a_l \psi \theta(m_{[1]}^\psi \otimes c_l) = m, \quad (28)$$

for all $m \in M \in \mathcal{M}(\psi)_A^C$. Taking $M = C \otimes A$, $m = d \otimes 1$, one obtains (27).

For all $n \in N \in \mathcal{M}_A$ and $d \in C$, one has

$$\begin{aligned} v_{G(N)}(G(\zeta_N)(n \otimes d)) &= v_{G(N)}(\sum_l n a_l \otimes c_l \otimes d) \\ &= \sum_l (n a_l \otimes c_{l(1)}) \theta(c_{l(2)} \otimes d) \\ &= \sum_l n a_l \theta(c_{l(2)} \otimes d) \psi \otimes c_{l(1)}^\psi \\ (18) \quad &= \sum_l n a_l \theta(c_l \otimes d_{(1)}) \otimes d_{(2)} \end{aligned}$$

and (2) can be written as

$$n \otimes d = \sum_l n a_l \theta(c_l \otimes d_{(1)}) \otimes d_{(2)}, \quad (29)$$

for all $n \in N \in \mathcal{M}_A$ and $d \in C$. Taking $N = A$ and $n = 1$, one obtains

$$1 \otimes d = \sum_l a_l \theta(c_l \otimes d_{(1)}) \otimes d_{(2)}.$$

Applying ϵ_C to the second factor, one finds (26).

Conversely, suppose that $\theta \in V_1$ and $z \in W_1$ satisfy (26) and (27). (27) implies (28), and (26) implies (29). Let $v = \alpha^{-1}(\theta)$, $\zeta = \beta^{-1}(z)$. Then (1-2) hold, and (F, G) is a Frobenius pair. \square

In [13] it is shown that if (H, A, C) is a Doi-Hopf structure, A is faithfully flat as a k -module, and C is projective as a k -module, then C is finitely generated. The next proposition shows that, in fact, one does not need the assumption that C is projective.

Proposition 3.5 *Let (A, C, ψ) be a right-right entwining structure. If (F, G) is a Frobenius pair, then $A \otimes C$ is finitely generated and projective as a left A -module.*

Proof. Let θ and $z = \sum_l a_l \otimes c_l$ be as in Theorem 3.4. Then for all $d \in C$,

$$\begin{aligned}
1 \otimes d &= \psi(d \otimes 1) \\
&= \psi(d_{(1)} \otimes \varepsilon(d_{(2)})1) \\
(27) \quad &= \sum_l \psi(d_{(1)} \otimes a_{l\psi} \theta(d_{(2)}^\Psi \otimes c_l)) \\
(7) \quad &= \sum_l a_{l\psi} \theta(d_{(2)}^\Psi \otimes c_l)_{\Psi'} \otimes d_{(1)}^{\Psi\Psi'} \\
(9) \quad &= \sum_l a_{l\psi} \theta((d^\Psi)_{(2)} \otimes c_l)_{\Psi'} \otimes (d^\Psi)_{(1)}^{\Psi'} \\
(18) \quad &= \sum_l a_{l\psi} \theta(d^\Psi \otimes c_{l(1)}) \otimes c_{l(2)}.
\end{aligned}$$

Write $c_{l(1)} \otimes c_{l(2)} = \sum_{j=1}^{m_l} c_{lj} \otimes c'_{lj}$ and for all l, j consider the map

$$\sigma_{lj} : A \otimes C \rightarrow A, \quad \sigma_{lj}(a \otimes d) = a a_{l\psi} \theta(d^\Psi \otimes c_{lj}).$$

Then for all $a \in A$ and $d \in C$,

$$a \otimes d = \sum_{l,j} \sigma_{lj}(a \otimes d)(1 \otimes c'_{lj}),$$

so $\{\sigma_{lj}, 1 \otimes c'_{lj} \mid l = 1, \dots, n, j = 1, \dots, m_l\}$ is a finite dual basis for $A \otimes C$ as a left A -module. \square

In some situations, one can conclude that C is finitely generated and projective as a k -module.

Corollary 3.6 *Let (A, C, ψ) be a right-right entwining structure, and assume that (F, G) is a Frobenius pair.*

- 1) *If A is faithfully flat as a k -module, then C is finitely generated as a k -module.*
- 2) *If A is commutative and faithfully flat as a k -module, then C is finitely generated projective as a k -module.*
- 3) *If k is a field, then C is finite dimensional as a k -vector space.*
- 4) *If $A = k$, then C is finitely generated projective as a k -module.*

Proof. 1) With notation as in Proposition 3.5, let M be the k -module generated by the c'_{lj} . Then for all $d \in C$,

$$1 \otimes d = \sum_{l,j} \sigma_{lj}(1 \otimes d) \otimes c'_{lj} \in A \otimes M.$$

Since A is faithfully flat, it follows that $d \in M$, hence $M = C$ is finitely generated.

2) From descent theory: if a k -module becomes finitely generated and projective after a faithfully flat commutative base extension, then it is itself finitely generated and projective.

3) Follows immediately from 1): since k is a field, A is faithfully flat as a k -module, and C is projective as a k -module.

4) Follows immediately from 2). \square

Now we want to recover [13, Theorem 2.4] and [3, Proposition 3.5]. Assume that C is finitely generated and projective as a k -module, and let $\{d_i, d_i^* \mid i = 1, \dots, m\}$ be a finite dual basis for C . Then $C^* \otimes A$ can be made into an object of ${}_A \mathcal{M}(\psi)_A^C$ as follows: for all $a, b, b' \in A$, $c^* \in C^*$,

$$b(c^* \otimes a)b' = \sum_i \langle c^*, d_i^\Psi \rangle d_i^* \otimes b_\psi a b', \quad (30)$$

$$\rho^r(c^* \otimes a) = \sum_i d_i^* * c^* \otimes a_\psi \otimes d_i^\Psi. \quad (31)$$

This can be checked directly. An explanation for this at first sight artificial structure is given in Section 5. We now give alternative descriptions for V and W . Recall from [10] that there are many possibilities to describe V . As we have seen, a natural transformation $v \in V$ is completely determined by θ . Nevertheless, the maps \bar{v} , \underline{v} , $\bar{\lambda}$ or $\underline{\lambda}$ (with notation as in Proposition 3.1) are possible alternatives. The map $\bar{\lambda} : C \otimes A \otimes C \rightarrow A$ induces $\bar{\phi} : A \otimes C \rightarrow C^* \otimes A \cong \text{Hom}(C, A)$. This is the map we need. At some place it is convenient to use $C^* \otimes A$ as the image space, at some other we prefer $\text{Hom}(C, A)$. Note that $\bar{\phi}$ is given by

$$\bar{\phi}(a \otimes c)(d) = \bar{\lambda}(d \otimes a \otimes c) = \underline{\lambda}(a_\psi \otimes d^\psi \otimes c) = a_\psi \theta(d^\psi \otimes c),$$

or

$$\bar{\phi}(a \otimes c) = \sum_i d_i^* \otimes a_\psi \theta(d_i^\psi \otimes c). \quad (32)$$

It turns out that $\bar{\phi}$ is a morphism in ${}_A \mathcal{M}_A^C$. More specifically, one has

Proposition 3.7 *Let (A, C, ψ) be a right-right entwining structure. If C is a finitely generated and projective k -module, then*

$$V \cong V_1 \cong V_2 = \text{Hom}_{AA}^{kC}(A \otimes C, C^* \otimes A).$$

The isomorphism is $\alpha_1 : V_1 \rightarrow V_2$, with $\alpha_1(\theta) = \bar{\phi}$ given by (32). The inverse of α_1 is

$$\alpha_1^{-1}(\bar{\phi})(d \otimes c) = \bar{\phi}(1 \otimes c)(d). \quad (33)$$

Proof. We first show that $\bar{\phi} \in V_2$. For all $a, b \in A$ and $c \in C$, we have

$$\begin{aligned} b\bar{\phi}(a \otimes c) &= b\left(\sum_i d_i^* \otimes a_\psi \theta(d_i^\psi \otimes c)\right) \\ &= \sum_{i,j} \langle d_i^*, d_j^\psi \rangle d_j^* \otimes b_\psi a_\psi \theta(d_i^\psi \otimes c) \\ &= \sum_j d_j^* \otimes b_\psi a_\psi \theta(d_j^{\psi\psi} \otimes c) \\ (7) \quad &= \sum_j d_j^* \otimes (ba)_\psi \theta(d_j^\psi \otimes c) \\ &= \bar{\phi}(ba \otimes c) = \bar{\phi}(b(a \otimes c)), \end{aligned}$$

proving that $\bar{\phi}$ is left A -linear. It is also right A -linear because

$$\begin{aligned} \bar{\phi}(a \otimes c)b &= \sum_i d_i^* \otimes a_\psi \theta(d_i^\psi \otimes c)b \\ (17) \quad &= \sum_i d_i^* \otimes a_\psi b_{\psi'\psi} \theta(d_i^{\psi\psi'} \otimes c^{\psi'}) \\ (7) \quad &= \sum_i d_i^* \otimes (ab_{\psi'})_\psi \theta(d_i^\psi \otimes c^{\psi'}) \\ &= \bar{\phi}(ab_{\psi'} \otimes c^{\psi'}) = \bar{\phi}(a \otimes cb). \end{aligned}$$

Notice that the dual basis for C satisfies the following equality (the proof is left to the reader):

$$\sum_i \Delta(d_i) \otimes d_i^* = \sum_{i,j} d_i \otimes d_j \otimes d_i^* * d_j^*. \quad (34)$$

Using this equality one computes

$$\begin{aligned}
\rho^r(\bar{\phi}(a \otimes c)) &= \rho^r\left(\sum_i d_i^* \otimes a_\Psi \theta(d_i^\Psi \otimes c)\right) \\
(31) \quad &= \sum_{i,j} d_j^* * d_i^* \otimes \left(a_\Psi \theta(d_i^\Psi \otimes c)\right)_\Psi \otimes d_j^\Psi \\
(34) \quad &= \sum_i d_i^* \otimes \left(a_\Psi \theta(d_{i(2)}^\Psi \otimes c)\right)_\Psi \otimes d_{i(1)}^\Psi \\
(7) \quad &= \sum_i d_i^* \otimes a_{\Psi\Psi} \theta(d_{i(2)}^\Psi \otimes c)_{\Psi'} \otimes d_{i(1)}^{\Psi\Psi'} \\
(9) \quad &= \sum_i d_i^* \otimes a_\Psi \theta((d_i^\Psi)_{(2)} \otimes c)_{\Psi'} \otimes (d_i^\Psi)_{(1)}^{\Psi'} \\
(18) \quad &= \sum_i d_i^* \otimes a_\Psi \theta(d_i^\Psi \otimes c_{(1)}) \otimes c_{(2)} \\
&= \bar{\phi}(a \otimes c_{(1)}) \otimes c_{(2)}.
\end{aligned}$$

This proves that $\bar{\phi}$ is right C -colinear. Conversely, given $\bar{\phi} \in V_2$, first one needs to show that $\theta = \alpha_1^{-1}(\bar{\phi}) \in V_1$. It is now more convenient to work with $\text{Hom}(C, A)$ rather than $C^* \otimes A$. For $f \in \text{Hom}(C, A)$, $b, b' \in A$, (30) can be rewritten as

$$(bfb')(c) = b_\Psi f(c^\Psi) b'. \quad (35)$$

Take any $c, d \in C$, $a \in A$ and compute

$$\begin{aligned}
\theta(c \otimes d)a &= \left(\bar{\phi}(1 \otimes d)(c)\right)a \\
(35) \quad &= \left(\bar{\phi}(1 \otimes d)a\right)(c) \\
(\bar{\phi} \text{ is right } A\text{-linear}) &= \bar{\phi}(a_\Psi \otimes d^\Psi)(c) \\
(\bar{\phi} \text{ is left } A\text{-linear}) &= \left(a_\Psi \bar{\phi}(1 \otimes d^\Psi)\right)(c) \\
(35) \quad &= a_{\Psi\Psi} \left(\bar{\phi}(1 \otimes d^\Psi)(c^\Psi)\right) \\
&= a_{\Psi\Psi} \bar{\phi}(c^\Psi \otimes d^\Psi).
\end{aligned}$$

This proves that θ satisfies (17). Before proving (18), we look at the right C -coaction ρ^r on $f = c^* \otimes a \in \text{Hom}(C, A) \cong C^* \otimes A$. Write $\rho^r(f) = f_{[0]} \otimes f_{[1]} \in \text{Hom}(C, A) \otimes C$. Using (31), we find, for all $c \in C$,

$$\begin{aligned}
f_{[0]}(c) \otimes f_{[1]} &= \sum_i \langle d_i^* * c^*, c \rangle a_\Psi d_i^\Psi \\
&= \sum_i \langle d_i^*, c_{(1)} \rangle \langle c^*, c_{(2)} \rangle a_\Psi d_i^\Psi \\
&= \langle c^*, c_{(2)} \rangle a_\Psi c_{(1)}^\Psi \\
&= \Psi(c_{(1)} \otimes f(c_{(2)})).
\end{aligned}$$

This means that for all $f \in \text{Hom}(C, A)$

$$f_{[0]}(c) \otimes f_{[1]} = \Psi(c_{(1)} \otimes f(c_{(2)})). \quad (36)$$

This can be used to show that θ satisfies (18). Explicitly,

$$\theta(c_{(2)} \otimes d)_\Psi \otimes c_{(1)}^\Psi = \Psi(c_{(1)} \otimes \theta(c_{(2)} \otimes d))$$

$$\begin{aligned}
(36) \quad &= \bar{\phi}(1 \otimes d)_{[0]}(c) \otimes \bar{\phi}(1 \otimes d)_{[1]} \\
(\bar{\phi} \text{ is right } C\text{-colinear}) \quad &= \bar{\phi}(1 \otimes d_{(1)})(c) \otimes d_{(2)} \\
&= \theta(c \otimes d_{(1)}) \otimes d_{(2)}.
\end{aligned}$$

It remains to be shown that α_1 and α_1^{-1} are inverses of each other. First take $\theta \in V_1$. Then for all $c, d \in C$,

$$\begin{aligned}
((\alpha_1^{-1} \circ \alpha_1)(\theta))(d \otimes c) &= \alpha_1(\theta)(1 \otimes c)(d) \\
&= \sum_i \langle d_i^*, d \rangle 1_\Psi \theta(d_i^\Psi \otimes c) \\
&= \theta(d \otimes c).
\end{aligned}$$

Finally, for $\bar{\phi} \in V_2$, $a \in A$ and $c, d \in C$:

$$\begin{aligned}
((\alpha_1 \circ \alpha_1^{-1})(\bar{\phi}))(a \otimes c)(d) &= \sum_i \langle d_i^*, d \rangle a_\Psi \alpha_1^{-1}(\bar{\phi})(d_i^\Psi \otimes c) \\
&= \sum_i \langle d_i^*, d \rangle a_\Psi \bar{\phi}(1 \otimes c)(d_i^\Psi) \\
&= a_\Psi \bar{\phi}(1 \otimes c)(d^\Psi) \\
&= (a\bar{\phi}(1 \otimes c))(d) = \bar{\phi}(a \otimes c)(d).
\end{aligned}$$

□

Now we give an alternative description for W_2 .

Proposition 3.8 *Let C be finitely generated and projective as a k -module. Then*

$$W \cong W_1 \cong W_2 = \text{Hom}_{AA}^{\text{kC}}(C^* \otimes A, A \otimes C).$$

The isomorphism $\beta_1 : W_1 \rightarrow W_2$ is given by $\beta_1(z) = \phi$ with

$$\phi(c^* \otimes a) = \sum_l a_l a_\Psi \otimes \langle c^*, c_{l(2)} \rangle c_{l(1)}^\Psi,$$

and the inverse of β_1 is given by

$$\beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1). \quad (37)$$

Proof. We have to show that $\beta_1(z) = \phi$ is left and right A -linear and right C -colinear. For all $c^* \in C^*$ and $a, b \in A$,

$$\begin{aligned}
\Phi(c^* \otimes ab) &= \sum_l a_l (ab)_\Psi \otimes \langle c^*, c_{j(2)} \rangle c_{j(1)}^\Psi \\
(7) \quad &= \sum_l a_l a_\Psi b_\Psi \otimes \langle c^*, c_{j(2)} \rangle c_{j(1)}^{\Psi\Psi} \\
&= \left(\sum_l a_l a_\Psi \otimes \langle c^*, c_{j(2)} \rangle c_{j(1)}^\Psi \right) b \\
&= \Phi(c^* \otimes a)b,
\end{aligned}$$

proving that Φ is right A -linear. The proof of left A -linearity goes as follows:

$$\begin{aligned}
\Phi(b(c^* \otimes a)) &= \sum_i \Phi(\langle c^*, d_i^\Psi \rangle d_i^* \otimes b_\Psi a) \\
&= \sum_{i,l} \langle c^*, d_i^\Psi \rangle a_l (b_\Psi a)_\Psi \otimes \langle d_i^*, c_{l(2)} \rangle c_{l(1)}^\Psi \\
(7) \quad &= \sum_{i,l} \langle c^*, d_i^\Psi \rangle a_l b_\Psi a_{\Psi'} \otimes \langle d_i^*, c_{l(2)} \rangle c_{l(1)}^{\Psi\Psi'} \\
&= \sum_l \langle c^*, c_{l(2)}^\Psi \rangle a_l b_\Psi a_{\Psi'} \otimes c_{l(1)}^{\Psi\Psi'} \\
(9) \quad &= \sum_l \langle c^*, (c_l^\Psi)_{(2)} \rangle a_l b_\Psi a_{\Psi'} \otimes (c_l^\Psi)_{(1)}^{\Psi'} \\
(23) \quad &= \sum_l \langle c^*, (c_{l(2)}) \rangle b a_l a_{\Psi'} \otimes (c_{l(1)}^{\Psi'}) \\
&= b\Phi(c^* \otimes a).
\end{aligned}$$

Next one needs to show that Φ is right C -colinear. Using (36), one finds

$$\begin{aligned}
\bar{\Phi}((c^* \otimes a)_{[0]}) \otimes (c^* \otimes a)_{[1]} &= \sum_i \bar{\Phi}(d_i^* * c^* \otimes a_\Psi) \otimes d_i^\Psi \\
&= \sum_{i,l} a_l a_\Psi \otimes \langle d_i^* * c^*, c_{l(2)} \rangle c_{l(1)}^\Psi \otimes d_i^\Psi \\
&= \sum_{i,l} a_l a_\Psi \otimes \langle d_i^*, c_{l(2)} \rangle \langle c^*, c_{l(3)} \rangle c_{l(1)}^\Psi \otimes d_i^\Psi \\
&= \sum_l a_l a_\Psi \otimes \langle c^*, c_{l(3)} \rangle c_{l(1)}^\Psi \otimes c_{l(2)}^\Psi \\
(9) \quad &= \sum_l a_l a_\Psi \otimes \langle c^*, c_{l(2)} \rangle (c_l^\Psi)_{(1)} \otimes (c_l^\Psi)_{(2)} \\
&= \rho^r(\bar{\Phi}(c^* \otimes a)).
\end{aligned}$$

Conversely, let $\phi \in W_2$ and put $z = \phi(\varepsilon \otimes 1) = \sum_l a_l \otimes c_l$. Using (30), we see that $a(\varepsilon \otimes 1) = (\varepsilon \otimes 1)a$, for all $a \in A$, hence $az = a\phi(\varepsilon \otimes 1) = \phi(a(\varepsilon \otimes 1)) = \phi((\varepsilon \otimes 1)a) = \phi(\varepsilon \otimes 1)a = za$, and $z \in W_1$.

Take $z = \sum_l a_l \otimes c_l \in W_1$. Then

$$\beta_1^{-1}(\beta_1(z)) = \sum_l a_l 1_\Psi \otimes \langle \varepsilon, c_{l(2)} \rangle c_{l(1)}^\Psi = z.$$

Finally, take $\phi \in W_2$, and write $\beta_1^{-1}(\phi) = \phi(\varepsilon \otimes 1) = \sum_l a_l \otimes c_l$. $C^* \otimes A$ and $A \otimes C$ are right C -comodules and left C^* -modules. Since ϕ is right A -linear, right C -colinear and left C^* -linear,

$$\begin{aligned}
\phi(c^* \otimes a) &= \phi(c^* \otimes 1)a = (c^* \cdot \phi(\varepsilon \otimes 1))a = (c^* \cdot (\sum_l a_l \otimes c_l))a \\
&= \langle c^*, c_{l(2)} \rangle \sum_l a_l a_\Psi \otimes c_{l(1)}^\Psi = \beta_1(z)(c^* \otimes a)
\end{aligned}$$

and it follows that $\phi = \beta_1(z) = \beta_1(\beta_1^{-1}(\phi))$, as required. \square

Suppose that C is finitely generated and projective as a k -module. From Proposition 3.7, it follows that F is separable if and only if there exists a map $\bar{\phi} \in V_2 = \text{Hom}_{A,A}^C(A \otimes C, C^* \otimes A)$ such that $\bar{\phi}(1 \otimes c_{(2)})(c_{(1)}) = \varepsilon(c)1$, for all $c \in C$. In the Doi-Hopf case, this implies the Maschke Theorem in [12]. Now we apply the same procedure to determine when (F, G) is a Frobenius pair.

Theorem 3.9 Consider an entwining structure (A, C, ψ) , and assume that C is finitely generated projective as a k -module. Let $F : \mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}_A$ be the functor forgetting the C -coaction, and $G = \bullet \otimes C$ be its right adjoint. Then the following statements are equivalent:

- 1) (F, G) is a Frobenius pair.
- 2) There exist $z = \sum a_l \otimes c_l \in W_1$ and $\theta \in V_1$ such that the maps

$$\phi : C^* \otimes A \rightarrow A \otimes C \quad \text{and} \quad \bar{\phi} : A \otimes C \rightarrow C^* \otimes A,$$

given by

$$\phi(c^* \otimes a) = \sum_l a_l a_\psi \otimes \langle c^*, c_{l(2)} \rangle c_{l(1)}^\psi, \quad (38)$$

$$\bar{\phi}(a \otimes c) = \sum_i d_i^* \otimes a_\psi \theta(d_i^\psi \otimes c), \quad (39)$$

are inverses of each other.

- 3) $C^* \otimes A$ and $A \otimes C$ are isomorphic as objects in ${}_A \mathcal{M}(\psi)_A^C$.

Proof. 1) \Rightarrow 2). Let $z \in W_1$ and $\theta \in V_1$ be as in Theorem 3.4. Then $\phi = \beta_1(z)$ and $\bar{\phi} = \alpha_1(\theta)$ are morphisms in ${}_A \mathcal{M}(\psi)_A^C$, and

$$\begin{aligned} \bar{\phi}(\phi(\varepsilon \otimes 1)) &= \bar{\phi}(z) = \sum_{i,l} d_i^* \otimes a_{l\psi} \theta(d_i^\psi \otimes c_l) \\ (27) \quad &= \sum_{i,l} d_i^* \otimes \varepsilon(d_i) 1 = \varepsilon \otimes 1. \end{aligned}$$

The fact that ϕ and $\bar{\phi}$ are right A -linear and left C^* -linear implies that $\bar{\phi} \circ \phi = I_{C^* \otimes A}$. Similarly, for all $c \in C$,

$$\begin{aligned} \phi(\bar{\phi}(1 \otimes c)) &= \phi\left(\sum_i d_i^* \otimes \theta(d_i \otimes c)\right) \\ &= \sum_{i,l} a_l \theta(d_i \otimes c)_\psi \otimes \langle d_i^*, c_{l(2)} \rangle c_{l(1)}^\psi \\ &= \sum_l a_l \theta(c_{l(2)} \otimes c)_\psi \otimes c_{l(1)}^\psi \\ (18) \quad &= \sum_l a_l \theta(c \otimes c_{(1)}) \otimes c_{(2)} \\ (26) \quad &= \varepsilon(c_{(1)}) 1 \otimes c_{(2)} \\ &= 1 \otimes c. \end{aligned}$$

Since ϕ and $\bar{\phi}$ are left A -linear, $\phi \circ \bar{\phi} = I_{A \otimes C}$.

2) \Rightarrow 3). Obvious, since ϕ and $\bar{\phi}$ are in ${}_A \mathcal{M}(\psi)_A^C$.

3) \Rightarrow 1). Let $\phi : C^* \otimes A \rightarrow A \otimes C$ be the connecting isomorphism, and put $z = \phi(\varepsilon \otimes 1) = \sum_l a_l \otimes c_l \in W_1$, $\theta = \alpha_1^{-1}(\phi^{-1}) \in V_1$. Applying (32) and (37), one finds

$$\varepsilon \otimes 1 = \phi^{-1}(\phi(\varepsilon \otimes 1)) = \sum_i d_i^* \otimes a_{i\psi} \theta(d_i^\psi \otimes c_i).$$

Evaluating this equality at $c \in C$, one obtains (27). For all $c \in C$,

$$1 \otimes c = \phi(\phi^{-1}(1 \otimes c)) = \sum_l a_l \theta(c_l \otimes c_{(1)}) \otimes c_{(2)}.$$

Applying ε to the second factor, one finds (26). Theorem 3.4 implies that (F, G) is a Frobenius pair. \square

Remark 3.10 Recently M. Takeuchi observed that entwined modules can be viewed as comodules over certain corings. This observation has been exploited in [5] to derive some properties of coring counterparts of functors F and G . It is quite clear that the procedure applied in Section 1 to extension and restriction of scalars can be adapted to functors associated to corings, leading to a generalization of the results in this Section. This will be the subject of a future publication.

4 The functor forgetting the A -action

Again, let (A, C, ψ) be a right-right entwining structure. The functor $G' : \mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}^C$ forgetting the A -action has a left adjoint F' . The unit μ and the counit η of the adjunction are given at the end of Section 2.

Lemma 4.1 *Let $M \in {}_A\mathcal{M}(\psi)_A^C$, $N \in {}^C\mathcal{M}(\psi)_A^C$. Then $F'G'(M) \in {}_A\mathcal{M}(\psi)_A^C$ and $G'F' \in {}^C\mathcal{M}(\psi)_A^C$. The left structures are given by*

$$a(m \otimes b) = am \otimes b \text{ and } \rho^l(n \otimes b) = \sum n_{[-1]} \otimes n_{[0]} \otimes b,$$

for all $a, b \in A$, $m \in M$, $n \in N$. Furthermore μ_M is left A -linear, and ν_N is left C -colinear.

Now write $V' = \underline{\text{Nat}}(G'F', 1_{\mathcal{M}^C})$, $W' = \underline{\text{Nat}}(1_C, F'G')$. Following the philosophy of the previous Sections, we give more explicit descriptions of V' and W' . We do not give detailed proofs, however, since the arguments are dual to the ones in the previous Section. Let

$$V'_1 = \{\vartheta \in (C \otimes A)^* \mid \vartheta(c_{(1)} \otimes a_\psi) c_{(2)}^\Psi = \vartheta(c_{(2)} \otimes a) c_{(1)}, \text{ for all } c \in C, a \in A\}. \quad (40)$$

Proposition 4.2 *The map $\alpha : V' \rightarrow V'_1$, $\alpha(v') = \varepsilon \circ \nu_C$ is an isomorphism.*

Proof. Details are left to the reader. Given $\vartheta \in V'$, for $N \in \mathcal{M}^C$, the natural map $\nu'_N : N \otimes A \rightarrow N$ is

$$\nu'_N(n \otimes a) = \sum \vartheta(n_{[1]} \otimes a) n_{[0]}$$

□

For any k -linear map $e : C \rightarrow A \otimes A$ and $c \in C$, we use the notation $e(c) = e^1(c) \otimes e^2(c)$ (summation understood). Let W'_1 be the k -submodule of $\text{Hom}(C, A \otimes A)$ consisting of maps e satisfying

$$e^1(c_{(1)}) \otimes e^2(c_{(1)}) \otimes c_{(2)} = e^1(c_{(2)})_\Psi \otimes e^2(c_{(2)})_\Psi \otimes c_{(1)}^{\Psi\Psi}, \quad (41)$$

$$e^1(c) \otimes e^2(c) a = a_\Psi e^1(c^\Psi) \otimes e^2(c^\Psi). \quad (42)$$

Proposition 4.3 *The map $\beta : W' \rightarrow W'_1$ given by*

$$\begin{aligned} \beta(\zeta') &= (\varepsilon \otimes I_A \otimes I_A) \circ \zeta'_{A \otimes C} \circ (\eta_A \otimes I_C) \\ &= (I_A \otimes \varepsilon \otimes I_A) \circ \zeta'_{C \otimes A} \circ (I_C \otimes \eta_A) \end{aligned}$$

is an isomorphism. Given $e \in W'_1$, $\zeta' = \beta^{-1}(e)$ is recovered from e as follows: for $M \in \mathcal{M}(\psi)_A^C$,

$$\zeta'_M(m) = \sum m_{[0]} e^1(m_{[1]}) \otimes e^2(m_{[1]}).$$

Proof. We show that β is well-defined, leaving other details to the reader. Consider a commutative diagram

$$\begin{array}{ccccc}
C & \xrightarrow{I_C \otimes \eta_A} & C \otimes A & \xrightarrow{\zeta'_{C \otimes A}} & C \otimes A \otimes A \\
\downarrow I_C & & \downarrow \Psi & & \downarrow \Psi \otimes I_A \\
C & \xrightarrow{\eta_A \otimes I_C} & A \otimes C & \xrightarrow{\zeta'_{A \otimes C}} & A \otimes C \otimes A
\end{array}$$

The map $\bar{\lambda} = \zeta'_{C \otimes A} \circ (I_C \otimes \eta_A)$ is left and right C -colinear. Write $\bar{\lambda}(c) = \sum_i c_i \otimes a_i \otimes a'_i$. Then

$$c_{(1)} \otimes \bar{\lambda}(c_{(2)}) = \sum_i c_{i(1)} \otimes c_{i(2)} \otimes a_i \otimes a'_i.$$

Applying ε to the second factor, one finds

$$c_{(1)} \otimes e(c_{(2)}) = \bar{\lambda}(c).$$

The right C -colinearity of $\bar{\lambda}$ implies that

$$\begin{aligned}
\bar{\lambda}(c_{(1)}) \otimes c_{(2)} &= \sum_i a_{i\Psi} \otimes a'_{i\Psi} \otimes c_i^{\Psi\Psi} \\
&= e^1(c_{(2)})_{\Psi} \otimes e^2(c_{(2)})_{\Psi} \otimes c_{(1)}^{\Psi\Psi},
\end{aligned}$$

and hence proves (41). To prove (42) note that $\underline{\lambda} = \zeta'_{A \otimes C} \circ (\eta_A \otimes I_C)$ is left and right A -linear, hence

$$\begin{aligned}
e^1(c) \otimes e^2(c)a &= (I_A \otimes \varepsilon \otimes I_A)(\zeta'_{A \otimes C}((1 \otimes c)a)) \\
&= (I_A \otimes \varepsilon \otimes I_A)(\zeta'_{A \otimes C}(a_{\Psi} \otimes c^{\Psi})) \\
&= a_{\Psi}(I_A \otimes \varepsilon \otimes I_A)(\zeta'_{A \otimes C}(1 \otimes c^{\Psi})) \\
&= a_{\Psi}e^1(c^{\Psi}) \otimes e^2(c^{\Psi}).
\end{aligned}$$

□

Proposition 4.4 *Let (A, C, Ψ) be a right-right entwining structure.*

1) $F' = \bullet \otimes A : \mathcal{M}^C \rightarrow \mathcal{M}(\Psi)_A^C$ is separable if and only if there exists $\vartheta \in V'_1$ such that for all $c \in C$,

$$\vartheta(c \otimes 1) = \varepsilon(c). \quad (43)$$

2) $G' : \mathcal{M}(\Psi)_A^C \rightarrow \mathcal{M}^C$ is separable if and only if there exists $e \in W'_1$ such that for all $c \in C$,

$$e^1(c)e^2(c) = \varepsilon(c)1. \quad (44)$$

3) (F', G') is a Frobenius pair if and only if there exist $\vartheta \in V'_1$ and $e \in W'_1$ such that

$$\varepsilon(c)1 = \vartheta(c_{(1)} \otimes e^1(c_{(2)}))e^2(c_{(2)}) \quad (45)$$

$$= \vartheta(c_{(1)}^{\Psi} \otimes e^2(c_{(2)}))e^1(c_{(2)})_{\Psi}. \quad (46)$$

Proof. We only prove 3). If (F', G') is a Frobenius pair, then there exist $v' \in V'$ and $\zeta' \in W'$ such that (1) and (2) hold. Take $\vartheta \in V'_1$ and $e \in W'_1$ corresponding to v' and ζ' and write down (1) applied to $n \otimes 1$ with $n \in N \in \mathcal{M}^C$,

$$n \otimes 1 = ((v'_N \otimes I_A) \circ \zeta'_{N \otimes A})(n \otimes 1) = \vartheta(n_{[1]} \otimes e^1(n_{[2]}))n_{[0]} \otimes e^1(n_{[2]}). \quad (47)$$

Taking $N = C$, $n = c$, and applying ε_C to the first factor, one obtains (45). Conversely, if $\vartheta \in V'_1$ and $e \in W'_1$ satisfy (45), then (47) is satisfied for all $N \in \mathcal{M}^C$, and (1) follows since $v'_N \otimes I_A$ and $\zeta'_{N \otimes A}$ are right A -linear. Now write down (2) applied to $m \in M \in \mathcal{M}(\psi)_A^C$,

$$m = (v'_{G'(M \otimes A)} \circ G'(\zeta'_M))(m) = \theta(m_{[1]}^\psi \otimes e^2(m_{[2]}))m_{[0]}e^1(m_{[2]})_\psi. \quad (48)$$

Take $M = C \otimes A$, $m = c \otimes 1$, and apply ε_C to the first factor. This gives (46). Conversely, if $\vartheta \in V'_1$ and $e \in W'_1$ satisfy (46), then application of (46) to the second and third factors in $m_{[0]} \otimes m_{[1]} \otimes 1$, and then ε_C to the second factor shows that (48) holds for all $M \in \mathcal{M}(\psi)_A^C$. Finally note that (48) is equivalent to (2). \square

Inspired by the results in the previous Section, we ask the following question: assuming (F', G') is a Frobenius pair, when is A finitely generated projective as a k -module. We give a partial answer in the next Proposition. We assume that ψ is bijective (cf. [2, Section 6]). In the Doi-Hopf case, this is true if the underlying Hopf algebra H has a twisted antipode. The inverse of ψ is then given by the formula

$$\psi^{-1}(a \otimes c) = c\bar{S}(a_{(1)}) \otimes a_{(0)}.$$

Proposition 4.5 *Let (A, C, ψ) be a right-right entwining structure. With notation as above, assume that (F', G') is a Frobenius pair. If there exists $c \in C$ such that $\varepsilon(c) = 1$, and if ψ is invertible, with inverse $\varphi = \psi^{-1} : A \otimes C \rightarrow C \otimes A$, then A is finitely generated and projective as a k -module.*

Proof. Observe first that (A, C, φ) is a left-left entwining structure. This means that (7-10) hold, but with A and C replaced by A^{op} and C^{cop} . In particular,

$$\varepsilon(c^\varphi)a_\varphi = \varepsilon(c)a, \quad (49)$$

$$a_\varphi \otimes \Delta(c^\varphi) = a_{\varphi\varphi} \otimes c_{(1)}^\varphi \otimes c_{(2)}^\varphi. \quad (50)$$

This can be seen as follows: rewrite (8) and (9) as commutative diagrams, reverse the arrows, and replace ψ by φ . Then we have (49) and (50) in diagram form. Now fix $c \in C$ such that $\varepsilon(c) = 1$. Then for all $a \in A$,

$$\begin{aligned} a = \varepsilon(c)a &= \varepsilon(c^\varphi)a_\varphi \\ (45) &= \vartheta((c^\varphi)_{(1)} \otimes e^1((c^\varphi)_{(2)}))e^2((c^\varphi)_{(2)})a_\varphi \\ (50) &= \vartheta(c_{(1)}^\varphi \otimes e^1(c_{(2)}^\varphi))e^2(c_{(2)}^\varphi)a_{\varphi\varphi} \\ (42) &= \vartheta(c_{(1)}^\varphi \otimes a_{\varphi\varphi\psi}e^1(c_{(2)}^{\varphi\psi}))e^2(c_{(2)}^{\varphi\psi}) \\ (\varphi = \psi^{-1}) &= \vartheta(c_{(1)}^\varphi \otimes a_\varphi e^1(c_{(2)}))e^2(c_{(2)}). \end{aligned}$$

Write $(I \otimes e)\Delta(c) = \sum_{i=1}^m c_i \otimes b_i \otimes a_i \in C \otimes A \otimes A$. For $i = 1, \dots, m$, define $a_i^* \in A^*$ by

$$\langle a_i^*, a \rangle = \vartheta(c_i^\varphi \otimes a_\varphi b_i).$$

Then $\{a_i, a_i^* \mid i = 1, \dots, m\}$ is a finite dual basis of A as a k -module. \square

From now on we assume that A is finitely generated and projective with finite dual basis $\{a_i, a_i^* \mid i = 1, \dots, m\}$. The proof of the next Lemma is straightforward, and therefore left to the reader.

Lemma 4.6 *Let (A, C, ψ) be a right-right entwining structure, and assume that A is finitely generated and projective as a k -module. Then $A^* \otimes C \in {}^C\mathcal{M}(\psi)_A^C$. The structure is given by the formulae*

$$(a^* \otimes c)b = \langle a^*, b_\psi a_i \rangle a_i^* \otimes c^\Psi, \quad (51)$$

$$\rho^r(a^* \otimes c) = a^* \otimes c_{(1)} \otimes c_{(2)}, \quad (52)$$

$$\rho^l(a^* \otimes c) = \langle a^*, a_{i\psi} \rangle c_{(1)}^\Psi \otimes a_i^* \otimes c_{(2)}^\Psi. \quad (53)$$

We now give alternative descriptions of V' and W' .

Proposition 4.7 *Let (A, C, ψ) be a right-right entwining structure, and assume that A is finitely generated and projective as a k -module. Then there is an isomorphism*

$$\beta_1 : W'_1 \rightarrow W'_2 = \text{Hom}_{kA}^{\text{CC}}(A^* \otimes C, C \otimes A),$$

$\beta_1(e) = \Omega$, with

$$\Omega(a^* \otimes c) = \langle a^*, e^1(c_{(2)})_\psi \rangle c_{(1)}^\Psi \otimes e^2(c_{(2)}).$$

The inverse of β_1 is given by $\beta_1^{-1}(\Omega) = e$ with

$$e(c) = \sum_i a_i \otimes (\epsilon_C \otimes I_A) \Omega(a^* \otimes c).$$

Proof. We first prove that β_1 is well-defined.

a) $\beta_1(e) = \Omega$ is right A -linear: for all $a^* \in A^*$, $c \in C$ and $b \in A$, we have

$$\begin{aligned} \Omega((a^* \otimes c)b) &= \sum_i \langle a^*, b_\psi a_i \rangle \Omega(a_i^* \otimes c^\Psi) \\ &= \sum_i \langle a^*, b_\psi a_i \rangle \langle a_i^*, e^1((c^\Psi)_{(2)})_\psi \rangle (c^\Psi)_{(1)}^\Psi \otimes e^2((c^\Psi)_{(2)}) \\ &= \langle a^*, b_\psi e^1((c^\Psi)_{(2)})_\psi \rangle (c^\Psi)_{(1)}^\Psi \otimes e^2((c^\Psi)_{(2)}) \\ (9) \quad &= \langle a^*, b_{\psi\psi'} e^1(c_{(2)}^\Psi)_\psi \rangle c_{(1)}^{\Psi'\Psi} \otimes e^2(c_{(2)}^\Psi) \\ (7) \quad &= \langle a^*, (b_\psi e^1(c_{(2)}^\Psi))_\psi \rangle c_{(1)}^\Psi \otimes e^2(c_{(2)}^\Psi) \\ (42) \quad &= \langle a^*, e^1(c_{(2)})_\psi \rangle c_{(1)}^\Psi \otimes e^2(c_{(2)})b \\ &= \Omega(a^* \otimes c)b. \end{aligned}$$

b) $\beta_1(e) = \Omega$ is right C -colinear: for all $a^* \in A^*$ and $c \in C$, we have

$$\begin{aligned} \rho^r(\Omega(a^* \otimes c)) &= \langle a^*, e^1(c_{(2)})_\psi \rangle \rho^r(c_{(1)}^\Psi \otimes e^2(c_{(2)})) \\ &= \langle a^*, e^1(c_{(2)})_\psi \rangle (c_{(1)}^\Psi)_{(1)} \otimes e^2(c_{(2)})_\psi \otimes (c_{(1)}^\Psi)_{(2)}^\Psi \\ (9) \quad &= \langle a^*, e^1(c_{(3)})_{\psi\psi'} \rangle c_{(1)}^{\Psi'} \otimes e^2(c_{(3)})_\psi \otimes c_{(2)}^{\Psi\Psi'} \\ (41) \quad &= \langle a^*, e^1(c_{(2)})_\psi \rangle c_{(1)}^{\Psi'} \otimes e^2(c_{(2)}) \otimes c_{(3)} \\ &= \Omega(a^* \otimes c_{(1)}) \otimes c_{(2)}. \end{aligned}$$

c) $\beta_1(e) = \Omega$ is left C -colinear: for all $a^* \in A^*$ and $c \in C$, we have

$$\rho^l(\Omega(a^* \otimes c)) = \langle a^*, e^1(c_{(2)})_\psi \rangle (c_{(1)}^\Psi)_{(1)} \otimes (c_{(1)}^\Psi)_{(2)} \otimes e^2(c_{(2)})$$

$$\begin{aligned}
(9) \quad &= \langle a^*, e^1(c_{(3)})_{\Psi\Psi'} \rangle c_{(1)}^{\Psi'} \otimes c_{(2)}^{\Psi} \otimes e^2(c_{(3)}) \\
&= \sum_i \langle a^*, a_{i\Psi} \rangle \langle a_i^*, e^1(c_{(3)})_{\Psi'} \rangle c_{(1)}^{\Psi'} \otimes c_{(2)}^{\Psi} \otimes e^2(c_{(3)}) \\
&= \sum_i \langle a^*, a_{i\Psi} \rangle c_{(1)}^{\Psi} \otimes \Omega(a_i^* \otimes c_{(2)}).
\end{aligned}$$

The proof that $\beta_1^{-1}(\Omega) = e$ satisfies (41) and (42) is left to the reader. The maps β_1 and β_1^{-1} are inverses of each other since

$$\begin{aligned}
\beta^{-1}(\beta(e))(c) &= \sum_i a_i \otimes (\varepsilon_C \otimes I_A) \langle a_i^*, e^1(c_{(2)})_{\Psi} \rangle c_{(1)}^{\Psi} \otimes e^2(c_{(2)}) \\
&= \sum_i a_i \otimes \langle a_i^*, e^1(c_{(2)})_{\Psi} \rangle \varepsilon_C(c_{(1)}^{\Psi}) \otimes e^2(c_{(2)}) \\
&= \langle a_i^*, e^1(c) \rangle a_i \otimes e^2(c) = e^1(c) \otimes e^2(c),
\end{aligned}$$

$$\begin{aligned}
\beta(\beta^{-1}(\omega))(a^* \otimes c) &= \langle a^*, (a_i)_{\Psi} \rangle c_{(1)}^{\Psi} \otimes (\varepsilon_C \otimes I_A) \Omega(a_i^* \otimes c_{(2)}) \\
&= (I_C \otimes \varepsilon_C \otimes I_A) (\langle a^*, (a_i)_{\Psi} \rangle c_{(1)}^{\Psi} \otimes \Omega(a_i^* \otimes c_{(2)})) \\
&= (I_C \otimes \varepsilon_C \otimes I_A) \rho^l(\Omega(a^* \otimes c)) \\
&= \Omega(a^* \otimes c).
\end{aligned}$$

At the last step, we used that for all $c \in C$ and $a \in A$,

$$(I_C \otimes \varepsilon_C \otimes I_A) \rho^l(c \otimes a) = c \otimes a.$$

□

Proposition 4.8 *Let (A, C, Ψ) be a right-right entwining structure. If A is a finitely generated projective k -module, then the map*

$$\alpha_1 : V'_1 \rightarrow V'_2 = \text{Hom}_{kA}^{\text{CC}}(C \otimes A, A^* \otimes C),$$

defined by $\alpha_1(\vartheta) = \overline{\Omega}$, with

$$\overline{\Omega}(c \otimes a) = \langle \vartheta, c_{(1)} \otimes a_{\Psi} a_i \rangle a_i^* \otimes c_{(2)}^{\Psi}$$

is an isomorphism. The inverse of α_1 is given by $\alpha_1^{-1}(\overline{\Omega}) = \vartheta$ with

$$\vartheta(c \otimes a) = \langle \overline{\Omega}(c \otimes a), 1_A \otimes \varepsilon_C \rangle.$$

Proof. We first show that α_1 is well-defined. Take $\vartheta \in V'_1$, and let $\alpha_1(\vartheta) = \overline{\Omega}$.

a) $\overline{\Omega}$ is right A -linear since for all $a, b \in A$ and $c \in C$,

$$\begin{aligned}
\overline{\Omega}(c \otimes a)b &= \sum_{i,j} \langle \vartheta, c_{(1)} \otimes a_{\Psi} a_i \rangle \langle a_i^*, b_{\Psi} a_j \rangle a_j^* \otimes c_{(2)}^{\Psi\Psi} \\
&= \sum_j \langle \vartheta, c_{(1)} \otimes a_{\Psi} b_{\Psi} a_j \rangle a_j^* \otimes c_{(2)}^{\Psi\Psi} \\
(7) \quad &= \sum_j \langle \vartheta, c_{(1)} \otimes (ab)_{\Psi} a_j \rangle a_j^* \otimes c_{(2)}^{\Psi} \\
&= \overline{\Omega}(c \otimes ab).
\end{aligned}$$

b) $\overline{\Omega}$ is right C -colinear since for all $a \in A$ and $c \in C$,

$$\begin{aligned}
 \rho^r(\overline{\Omega}(c \otimes a)) &= \vartheta(c_{(1)} \otimes a_\Psi a_i) a_i^* \otimes (c_{(2)}^\Psi)_{(1)} \otimes (c_{(2)}^\Psi)_{(2)} \\
 (9) \quad &= \vartheta(c_{(1)} \otimes a_{\Psi\Psi} a_i) a_i^* \otimes c_{(2)}^\Psi \otimes c_{(3)}^\Psi \\
 &= \overline{\Omega}(c_{(1)} \otimes a_\Psi) \otimes c_{(2)}^\Psi.
 \end{aligned}$$

c) $\overline{\Omega}$ is left C -colinear since for all $a \in A$ and $c \in C$,

$$\begin{aligned}
 \rho^l(\overline{\Omega}(c \otimes a)) &= \sum_{i,j} \vartheta(c_{(1)} \otimes a_\Psi a_i) \langle a_i^*, a_{j\Psi} \rangle (c_{(2)}^\Psi)_{(1)}^\Psi \otimes a_j^* \otimes (c_{(2)}^\Psi)_{(2)} \\
 (9) \quad &= \sum_{i,j} \vartheta(c_{(1)} \otimes a_{\Psi\Psi'} a_i) \langle a_i^*, a_{j\Psi} \rangle c_{(2)}^{\Psi'\Psi} \otimes a_j^* \otimes c_{(3)}^\Psi \\
 &= \sum_j \vartheta(c_{(1)} \otimes a_{\Psi\Psi'} a_{j\Psi}) c_{(2)}^{\Psi'\Psi} \otimes a_j^* \otimes c_{(3)}^\Psi \\
 (7) \quad &= \sum_j \vartheta(c_{(1)} \otimes (a_\Psi a_j)_{\Psi'}) c_{(2)}^{\Psi'} \otimes a_j^* \otimes c_{(3)}^\Psi \\
 (40) \quad &= \sum_j \vartheta(c_{(2)} \otimes a_\Psi a_j) c_{(1)} \otimes a_j^* \otimes c_{(3)}^\Psi \\
 &= c_{(1)} \otimes \overline{\Omega}(c_{(2)} \otimes a).
 \end{aligned}$$

Conversely, given $\overline{\Omega}$, we have to show that $\alpha_1^{-1}(\overline{\Omega}) = \vartheta$ satisfies (40). Take any $c \otimes a \in C \otimes A$ and write $\overline{\Omega}(c \otimes a) = \sum_l b_l^* \otimes d_l \in A^* \otimes C$. Since $\overline{\Omega}$ is right and left C -colinear, we have

$$\begin{aligned}
 \overline{\Omega}(c_{(1)} \otimes a_\Psi) \otimes c_{(2)}^\Psi &= \sum_l b_l^* \otimes d_{l(1)} \otimes d_{l(2)}, \\
 c_{(1)} \otimes \overline{\Omega}(c_{(2)} \otimes a) &= \sum_l \langle b_l^*, a_{i\Psi} \rangle d_{l(1)}^\Psi \otimes a_i^* \otimes d_{l(2)}.
 \end{aligned}$$

Therefore we can compute

$$\begin{aligned}
 \vartheta(c_{(2)} \otimes a) c_{(1)} &= \langle \overline{\Omega}(c_{(2)} \otimes a), 1_A \otimes \varepsilon \rangle c_{(1)} \\
 &= \sum_l \langle b_l^*, a_{i\Psi} \rangle \langle a_i^*, 1 \rangle \langle \varepsilon, d_{l(2)} \rangle d_{l(1)}^\Psi \\
 &= \sum_l \langle b_l^* 1_\Psi \rangle d_l^\Psi \\
 &= \sum_l \langle b_l^* 1 \rangle d_l \\
 &= \langle \overline{\Omega}(c_{(1)} \otimes a_\Psi), 1_A \otimes \varepsilon_C \rangle c_{(2)}^\Psi \\
 &= \vartheta(c_{(1)} \otimes a_\Psi) c_{(2)}^\Psi.
 \end{aligned}$$

Thus (40) follows. Finally, we show that α_1 and α_1^{-1} are inverses of each other.

$$\alpha_1^{-1}(\alpha_1(\vartheta))(c \otimes a) = \langle \vartheta(c_{(1)} \otimes a_\Psi a_i) a_i^* \otimes c_{(2)}^\Psi, 1_A \otimes \varepsilon_C \rangle = \vartheta(c_{(1)} \otimes a a_i) \langle a_i^*, 1_A \rangle = \vartheta(c \otimes a).$$

We know that $\alpha_1(\alpha_1^{-1}(\overline{\Omega}))$ is right A -linear. Hence suffices it to show that

$$\alpha_1(\alpha_1^{-1}(\overline{\Omega}))(c \otimes 1) = c \otimes 1,$$

for all $c \in C$. From (51), we compute

$$\langle (a^* \otimes c)b, 1_A \otimes \varepsilon_C \rangle = \langle a^*, b \rangle \varepsilon(c) = \langle a^* \otimes c, b \otimes \varepsilon_C \rangle.$$

Now write $\overline{\Omega}(c \otimes 1) = \sum_r a_r^* \otimes c_r$ and compute

$$\begin{aligned} \alpha_1(\alpha_1^{-1}(\overline{\Omega}))(c \otimes 1) &= \langle \overline{\Omega}(c_{(1)} \otimes a_i), 1_A \otimes \varepsilon_C \rangle a_i^* \otimes c_{(2)} \\ &= \langle \overline{\Omega}(c_{(1)} \otimes 1), a_i \otimes \varepsilon_C \rangle a_i^* \otimes c_{(2)} \\ &= \overline{\Omega}(c \otimes 1)_{[0]}, a_i \otimes \varepsilon_C \rangle a_i^* \otimes \overline{\Omega}(c \otimes 1)_{[1]} \\ &= \sum_r \langle a_r^*, a_i \rangle \langle \varepsilon, c_{r(1)} \rangle a_i^* \otimes c_{r(2)} \\ &= \sum_r a_r^* \otimes c_r = \overline{\Omega}(c \otimes 1). \end{aligned}$$

□

Theorem 4.9 *Let (A, C, ψ) be a right-right entwining structure, and assume that A is finitely generated and projective as a k -module. With notation as above, we have the following properties:*

1) F' is separable if and only if there exists $\overline{\Omega} \in V'_2$ such that for all $c \in C$,

$$\langle \overline{\Omega}(c \otimes 1), 1_A \otimes \varepsilon_C \rangle = \varepsilon_C(c).$$

2) G' is separable if and only if there exists $\Omega \in W'_2$ such that for all $c \in C$,

$$\sum_i a_i(\varepsilon_C \otimes I_A)\Omega(a_i^* \otimes c) = \varepsilon_C(c)1.$$

3) The following assertions are equivalent:

a) (F', G') is a Frobenius pair.

b) There exist $e \in W'_1$, $\vartheta \in V'_1$ such that $\Omega = \beta_1(e)$ and $\overline{\Omega} = \alpha_1(\vartheta)$ are inverses of each other.

c) $A^* \otimes C$ and $C \otimes A$ are isomorphic objects in ${}^C\mathcal{M}(\psi)_A$.

Proof. We only prove $a) \Rightarrow b)$ in 3). First we show that Ω is a left inverse of $\overline{\Omega}$. Since $\Omega \circ \overline{\Omega}$ is right A -linear, it suffices to show that

$$\begin{aligned} \Omega(\overline{\Omega}(c \otimes 1)) &= \sum_i \vartheta(c_{(1)} \otimes a_i) \Omega(a_i^* \otimes c_{(2)}) \\ &= \sum_i \vartheta(c_{(1)} \otimes a_i) \langle a_i^*, e^1(c_{(3)})_\psi \rangle c_{(2)}^\psi \otimes e^2(c_{(3)}) \\ &= \vartheta(c_{(1)} \otimes e^1(c_{(3)})_\psi) c_{(2)}^\psi \otimes e^2(c_{(3)}) \\ (40) \quad &= \vartheta(c_{(2)} \otimes e^1(c_{(3)})_\psi) c_{(1)}^\psi \otimes e^2(c_{(3)}) \\ (45) \quad &= c \otimes 1. \end{aligned}$$

To show that Ω is a right inverse of $\overline{\Omega}$ we use that $\overline{\Omega} \circ \Omega$ is right C -colinear and conclude that it suffices to show that for all $c \in C$ and $a^* \in A^*$,

$$(I_{A^*} \otimes \varepsilon_C)(\overline{\Omega}(\Omega(a^* \otimes c))) = \varepsilon_C(c)a^*.$$

Both sides of the equation are in A^* , so the proof is completed if we show that both sides are equal when evaluated at an arbitrary $a \in A$. Observe that

$$\overline{\Omega}(\Omega(a^* \otimes c)) = \sum_i \langle a^*, e^1(c_{(2)})_{\Psi} \rangle \vartheta((c_{(1)}^{\Psi})_{(1)} \otimes e^2(c_{(2)})_{\Psi} a_i^*) a_i^* \otimes (c_{(1)}^{\Psi})_{(2)}^{\Psi}$$

hence

$$\begin{aligned} & (I_{A^*} \otimes \varepsilon_C)(\overline{\Omega}(\Omega(a^* \otimes c)))(a) \\ &= \langle a^*, e^1(c_{(2)})_{\Psi} \rangle \vartheta(c_{(1)}^{\Psi} \otimes e^2(c_{(2)})a) \\ (42) \quad &= \langle a^*, (a_{\Psi} e^1(c_{(2)}^{\Psi}))_{\Psi} \rangle \vartheta(c_{(1)}^{\Psi} \otimes e^2(c_{(2)}^{\Psi})) \\ (7) \quad &= \langle a^*, a_{\Psi} e^1(c_{(2)}^{\Psi})_{\Psi'} \rangle \vartheta(c_{(1)}^{\Psi'} \otimes e^2(c_{(2)}^{\Psi})) \\ (9) \quad &= \langle a^*, a_{\Psi} e^1((c^{\Psi})_{(2)})_{\Psi'} \rangle \vartheta((c^{\Psi})_{(1)}^{\Psi'} \otimes e^2((c^{\Psi})_{(2)})) \\ (46) \quad &= \langle a^*, a_{\Psi} \rangle \varepsilon(c^{\Psi}) = \langle a^*, a \rangle \varepsilon(c), \end{aligned}$$

as required. \square

5 The smash product

Let (B, A, R) be a factorization structure (sometimes also called a smash or twisted tensor product structure, cf. [27][22, pp. 299-300][16]). This means that A and B are k -algebras and that $R : A \otimes B \rightarrow B \otimes A$ is a k -linear map such that for all $a, c \in A, b, d \in B$,

$$R(ac \otimes b) = b_{Rr} \otimes a_r c_R, \quad (54)$$

$$R(a \otimes bd) = b_R d_r \otimes a_{Rr}, \quad (55)$$

$$R(a \otimes 1_B) = 1_B \otimes a, \quad (56)$$

$$R(1_A \otimes b) = b \otimes 1_A. \quad (57)$$

We use the notation $R(a \otimes b) = b_R \otimes a_R$. $B \#_R A$ is the k -module $B \otimes A$ with new multiplication

$$(b \# a)(d \# c) = b d_R \# a_R c.$$

$B \#_R A$ is an associative algebra with unit $1_B \# 1_A$ if and only if (54-57) hold. In this Section we want to examine when $B \#_R A / A$ and $B \#_R A / B$ are separable or Frobenius. This will be a direct application of the results in the second part of Section 1.

In Section 1, take $R = A, S = B \#_R A$. For $\bar{v} \in V_1 = \text{Hom}_{R,R}(S, R)$, define $\kappa : B \rightarrow A$ by

$$\kappa(b) = \bar{v}(b \# 1).$$

Then \bar{v} can be recovered from κ , since $\bar{v}(b \# a) = \kappa(b)a$. Furthermore

$$a \kappa(b) = a \bar{v}(b \# 1) = \bar{v}(b_R \# a_R) = \kappa(b_R) a_R$$

and we find that

$$V \cong V_1 \cong V_3 = \{\kappa : B \rightarrow A \mid a \kappa(b) = \kappa(b_R) a_R\}. \quad (58)$$

Now we simplify the description of $W \cong W_1 \subset (B \#_R A) \otimes_A (B \#_R A)$. Note that there is a k -module isomorphism

$$\gamma : (B \#_R A) \otimes_A (B \#_R A) \rightarrow B \otimes B \otimes A,$$

defined by

$$\begin{aligned}\gamma((b\#a) \otimes (d\#c)) &= b \otimes d_R \otimes a_{RC}, \\ \gamma^{-1}(b \otimes d \otimes c) &= (b\#1) \otimes (d\#c).\end{aligned}$$

Let $W_3 = \gamma(W_1) \subset B \otimes B \otimes A$. Take $e = b^1 \otimes b^2 \otimes a^2 \in B \otimes B \otimes A$ (summation implicitly understood). Then $e \in W_3$ if and only if (4) holds, for all $s = b\#1$ and $s = 1\#a$ with $b \in B$ and $a \in A$, if and only if

$$bb^1 \otimes b^2 \otimes a^2 = b^1 \otimes b^2 b_R \otimes a_R^2, \quad (59)$$

$$(b^1)_R \otimes (b^2)_r \otimes a_{Rr} a^2 = b^1 \otimes b^2 \otimes a^2 a, \quad (60)$$

for all $a \in A, b \in B$. This implies isomorphisms

$$W \cong W_1 \cong W_3 = \{e = b^1 \otimes b^2 \otimes a^2 \in B \otimes B \otimes A \mid (59) \text{ and } (60) \text{ hold}\}. \quad (61)$$

Using these descriptions of V and W , we find immediately that Theorem 1.2 takes the following form.

Theorem 5.1 *Let (B, A, R) be a factorization structure over a commutative ring k .*

1) $B\#_R A/A$ is separable (i.e. the restriction of scalars functor $G : \mathcal{M}_{B\#_R A} \rightarrow \mathcal{M}_A$ is separable) if and only if there exists $e = b^1 \otimes b^2 \otimes a^2 \in W_3$ such that

$$b^1 b^2 \otimes a^2 = 1_B \otimes 1_A \in B \otimes A. \quad (62)$$

2) $B\#_R A/A$ is split (i.e. the induction functor $F : \mathcal{M}_A \rightarrow \mathcal{M}_{B\#_R A}$ is separable) if and only if there exists $\kappa \in V_3$ such that

$$\kappa(1_B) = 1_A. \quad (63)$$

3) $B\#_R A/A$ is Frobenius (i.e. (F, G) is Frobenius pair) if and only if there exist $\kappa \in V_3, e \in W_3$ such that

$$(b^2)^R \otimes \kappa(b^1)_R a^2 = b^1 \otimes \kappa(b^2) a^2 = 1_B \otimes 1_A. \quad (64)$$

Theorem 1.7 can be reformulated in the same style. Notice that

$$\text{Hom}_R(S, R) = \text{Hom}_A(B\#_R A, A) \cong \text{Hom}(B, A).$$

$\text{Hom}(B, A)$ has the following $(A, B\#_R A)$ -bimodule structure (cf. (6)):

$$(cf(b\#a))(d) = cf(db)a,$$

for all $a, c \in C$ and $b, d \in B$. From Proposition 1.5, we deduce that

$$V \cong V_2 \cong V_4 = \text{Hom}_{A, B\#_R A}(B\#_R A, \text{Hom}(B, A)). \quad (65)$$

If B is finitely generated and projective as a k -module, then we find using Proposition 1.6

$$W \cong W_2 \cong W_4 = \text{Hom}_{A, B\#_R A}(\text{Hom}(B, A), B\#_R A). \quad (66)$$

Theorem 1.7 now takes the following form:

Theorem 5.2 Let (B, A, R) be a factorization structure over a commutative ring k , and assume that B is finitely generated and projective as a k -module. Let $\{b_i, b_i^* \mid i = 1, \dots, m\}$ be a finite dual basis for B .

1) $B\#_R A/A$ is separable if and only if there exists an $(A, B\#_R A)$ -bimodule map $\phi : \text{Hom}(B, A) \cong B^* \otimes A \rightarrow B\#_R A$ such that

$$\sum_i (b_i \# 1) \phi(b_i^* \# 1) = 1_B \otimes 1_A.$$

2) $B\#_R A/A$ is split if and only if there exists an $(A, B\#_R A)$ -bimodule map $\bar{\phi} : B\#_R A \rightarrow \text{Hom}(B, A)$ such that

$$\bar{\phi}(1_B \# 1_A)(1_B) = 1_A.$$

3) $B\#_R A/A$ is Frobenius if and only if $B^* \otimes A$ and $B\#_R A$ are isomorphic as $(A, B\#_R A)$ -bimodules. This is also equivalent to the existence of $\kappa \in V_3$, $e = b^1 \otimes b^2 \otimes a^2 \in W_3$ such that the maps

$$\phi : \text{Hom}(B, A) \rightarrow B\#_R A, \quad \phi(f) = f(b^1) b^2 \# a^2$$

and

$$\bar{\phi} : B\#_R A \rightarrow \text{Hom}(B, A), \quad \bar{\phi}(b \# a)(d) = \kappa(b d_R) a_R$$

are inverses of each other.

The same method can be applied to the extension $B\#_R A/B$. There are two ways to proceed: as above, but applying the left-handed version of Theorem 1.7 (left and right separable (resp. Frobenius) extension coincide). Another possibility is to use “op”-arguments. If $R : A \otimes B \rightarrow B \otimes A$ makes (B, A, R) into a factorization structure, then

$$\tilde{R} : B^{\text{op}} \otimes A^{\text{op}} \rightarrow A^{\text{op}} \otimes B^{\text{op}}$$

makes $(A^{\text{op}}, B^{\text{op}}, \tilde{R})$ into a factorization structure. It is not hard to see that there is an algebra isomorphism

$$(A^{\text{op}} \#_{\tilde{R}} B^{\text{op}})^{\text{op}} \cong B\#_R A.$$

Using the left-right symmetry again, we find that $B\#_R A/B$ is Frobenius if and only if $(A^{\text{op}} \#_{\tilde{R}} B^{\text{op}})^{\text{op}}/B$ is Frobenius if and only if $(A^{\text{op}} \#_{\tilde{R}} B^{\text{op}})/B^{\text{op}}$ is Frobenius, and we can apply Theorems 5.1 and 5.2. We invite the reader to write down explicit results.

Our final aim is to link the results in this Section to the ones in Section 3, at least in the case of finitely generated, projective B . Let (A, C, ψ) be a right-right entwining structure, with C finitely generated and projective, and put $B = (C^*)^{\text{op}}$. Let $\{c_i, c_i^* \mid i = 1, \dots, n\}$ be a dual basis for C . There is a bijective correspondence between right-right entwining structures (A, C, ψ) and smash product structures $(C^{*\text{op}}, A, R)$. R and ψ can be recovered from each other using the formulae

$$R(a \otimes c^*) = \sum_i \langle c^*, c_i^\psi \rangle c_i^* \otimes a_\psi, \quad \psi(c \otimes a) = \sum_i \langle (c_i^*)_R, c \rangle c_i \otimes a_R.$$

Moreover, there are isomorphisms of categories

$$\mathcal{M}(\psi)_A^C \cong \mathcal{M}_{B\#A} \quad \text{and} \quad {}_A \mathcal{M}(\psi)_A^C \cong {}_A \mathcal{M}_{B\#A}.$$

In particular, $B\#_R A$ can be made into an object of ${}_A \mathcal{M}(\psi)_A^C$, and this explains the structure on $C^* \otimes A$ used in Section 3. Combining Theorems 3.9 and 5.2, we find that the forgetful functor $\mathcal{M}(\psi)_A^C \rightarrow \mathcal{M}_A$ and its adjoint form a Frobenius pair if and only if $C^* \otimes A$ and $A \otimes C$ are isomorphic as $(A, (C^*)^{\text{op}} \#_R A)$ -bimodules, if and only if the extension $(C^*)^{\text{op}} \#_R A/A$ is Frobenius.

References

- [1] K.I. Beidar, Y. Fong and A. Stolin, On Frobenius algebras and the Yang-Baxter equation, *Trans. Amer. Math. Soc.* **349** (1997), 3823-3836.
- [2] T. Brzeziński, On modules associated to coalgebra-Galois extensions, *J. Algebra* **215** (1999), 290–317.
- [3] T. Brzeziński, Frobenius properties and Maschke-type theorems for entwined modules, *Proc. Amer. Math. Soc.*, to appear.
- [4] T. Brzeziński, Coalgebra-Galois extensions from the extension point of view, in “Hopf algebras and quantum groups”, S. Caenepeel and F. Van Oystaeyen (Eds.), Marcel Dekker, New York, to appear.
- [5] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties, preprint math.RA/0002105.
- [6] T. Brzeziński and P. M. Hajac, Coalgebra extensions and algebra coextensions of Galois type, *Comm. Algebra* **27** (1999), 1347-1367.
- [7] T. Brzeziński and S. Majid, Coalgebra bundles, *Comm. Math. Phys.* **191** (1998), 467-492.
- [8] S. Caenepeel, B. Ion and G. Militaru, The structure of Frobenius algebras and separable algebras, *K-theory*, to appear.
- [9] S. Caenepeel, B. Ion, G. Militaru, and Shenglin Zhu, Smash biproducts of algebras and coalgebras, *Algebras and Representation Theory* **3** (2000), 19-42.
- [10] S. Caenepeel, B. Ion, G. Militaru, and Shenglin Zhu, Separable functors for the category of Doi-Hopf modules, Applications, *Adv. Math.* **145** (1999), 239-290.
- [11] S. Caenepeel, B. Ion, G. Militaru, and Shenglin Zhu, Separable functors for the category of Doi-Hopf modules II, in “Hopf algebras and quantum groups”, S. Caenepeel and F. Van Oystaeyen (Eds.), Marcel Dekker, New York, to appear.
- [12] S. Caenepeel, G. Militaru, and S. Zhu, A Maschke type theorem for Doi-Hopf modules, *J. Algebra* **187** (1997), 388–412.
- [13] S. Caenepeel, G. Militaru, and S. Zhu, Doi-Hopf modules, Yetter-Drinfel’d modules and Frobenius type properties, *Trans. Amer. Math. Soc.* **349** (1997), 4311–4342.
- [14] S. Caenepeel, G. Militaru, and S. Zhu, Crossed modules and Doi-Hopf modules, *Israel J. Math.* **100** (1997), 221–247.
- [15] S. Caenepeel and Ş. Raianu, Induction functors for the Doi-Koppinen unified Hopf modules, in “Abelian groups and Modules”, A. Facchini and C. Menini (Eds.), Kluwer Academic Publishers, Dordrecht, 1995, p. 73-94.
- [16] A. Čap, H. Schichl and J. Vanžura. On twisted tensor product of algebras. *Commun. Algebra* **23** (1995), 4701–4735.
- [17] Y. Doi, Unifying Hopf modules, *J. Algebra* **153** (1992), 373-385.
- [18] L. Kadison, The Jones polynomial and certain separable Frobenius extensions, *J. Algebra* **186** (1996), 461-475.

- [19] L. Kadison, Separability and the twisted Frobenius bimodules, *Algebras and Representation Theory* **2** (1999), 397-414.
- [20] L. Kadison, “New examples of Fobenius extensions”, *University Lect. Series* bf 14, Amer. Math. Soc., Providence, 1999.
- [21] M. Koppinen, Variations on the smash product with applications to group-graded rings, *J. Pure Appl. Algebra* **104** (1995), 61-80.
- [22] S. Majid. *Foundation of Quantum Group Theory*. Cambridge University Press 1995.
- [23] C. Năstăsescu, M. van den Bergh, and F. van Oystaeyen, Separable functors applied to graded rings, *J. Algebra* **123** (1989), 397-413.
- [24] R. Pierce, Associative algebras, *Grad. Text in Math.* **88**, Springer Verlag, Berlin, 1982.
- [25] M. D. Rafael, Separable functors revisited, *Comm. in Algebra* **18** (1990), 1445-1459.
- [26] A. del Río, Categorical methods in graded ring theory, *Publ. Math.* **72** (1990), 489-531.
- [27] D. Tambara. The coendomorphism bialgebra of an algebra, *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **37** (1990), 425–456.